BERTINI THEOREMS FOR DIFFERENTIAL ALGEBRAIC GEOMETRY

JAMES FREITAG

Abstract. We study intersection theory for differential algebraic varieties. Particularly, we study families of differential hypersurface sections of arbitrary affine differential algebraic varieties over a differential field. We prove the differential analogue of Bertini's theorem, namely that for an arbitrary geometrically irreducible differential algebraic variety, generic hyperplane sections are geometrically irreducible and codimension one. We also study hypersurface sections in families. In the case of families of hyperplanes, we also establish smoothness results. Following the main theorem, several applications of this work relating to the definability of Kolchin polynomials and the definability of irreducibility in families of differential algebraic varieties are given.

Consider the following theorem from algebraic geometry:

Theorem. [4, 7.1, page 48] *Let Y, Z be irreducible algebraic varieties of dimensions r, s in $\mathbb{A}^n$ Then every irreducible component W of $Y \cap Z$ has dimension greater than or equal to $r + s - n$.*

This theorem fails for differential algebraic varieties embedded in affine space, as the following example shows. Developing the following counterexample to the intersection theorem involves a number of fairly technical results from differential algebra. The reader unfamiliar with these developments might simply take them for granted or even simply look at the conclusion of the example, after which we will give a detailed introduction to our results.

Example 0.1. (Ritt's example). We work in $\mathbb{A}^3$ over an ordinary differential field, k. Let $V = Z(f)$, where

$$f(x, y, z) = x^5 - y^5 + z(x\delta y - y\delta x)^2$$

In fact, though V is the zero set of an absolutely irreducible differential polynomial, it is not irreducible in the Kolchin topology. V has six components. Let μ_5 denote the set of fifth roots of unity. For each $\zeta \in \mu_5$, $x - \zeta y$ cuts out a subvariety of V. Note that

$$f = (x - \zeta y)\Big(\prod_{\eta \in \mu_5 \setminus \{\zeta\}} (x - \eta y) + z(\delta y(x - \zeta y) - y\delta(x - \zeta y))^2$$

[1]Supported by National Science Foundation grant DMS-1204510 and an American Mathematical Society Mathematical Research Communities grant.

is a preparation eqution for f with respect to $x-\zeta y$ [8, see chapter 4, section 13]. Further, one obtains the preparation congruence, $f = (x-\zeta y)(\prod_{\eta \in \mu_5 \backslash \{\zeta\}}(x-\eta y)$ modulo $[x-\zeta y]^2$, so by the Low Power Theorem [22, chapter 7] or [8, chapter 4, section 15] $[x-\zeta]$ is the ideal of a component of V. The general component is given by the saturation by the separant (with respect to some ranking) of $[f]$. For instance, one possible choice of ranking would yield $[f] : \frac{\partial f}{\partial \delta x}^{\infty} = \{g \mid \left(\frac{\partial f}{\partial \delta x}\right)^n g \in [f]$, for some $n \in \mathbb{N}\}$ as the ideal of the general component. By the component theorem [8, Theorem 5, page 185], these are the only components. Now we consider the differential algebraic variety W, which is the general component of V.

Establishing that $(0,0,0) \in W$ can be done by noting that f is differentially homogeneous and applying [7, Proposition 2] to show that all of the components are differentially homogeneous (thanks to Phyllis Cassidy for this observation). Now, consider $W \cap H$ where H is the hyperplane given by $z=0$. Take $(a,b,0) \in W$. Then $\prod_{\eta \in \mu_5}(a-\eta b) = 0$, so

$$a - \zeta b = 0 \tag{1}$$

for some $\zeta \in \mu_5$; fix this ζ for the remainder of the example. Now the problem is essentially to determine the intersection of the line given by $z=0$ and $x-\zeta y$ with the general component W, given by $[f] : \frac{\partial f}{\partial \delta x}^{\infty}$ (or equivalently saturating with respect to some other ranking).

For each $\eta \in \mu_5 \backslash \{\zeta\}$, consider new variables z_1, u_0, u_1, u_2, u_3 and let

$$G = u_0 z_1 + u_1 z_1^2 + u_2 z_1 \delta z_1 + u_3 (\delta z_1)^2.$$

Applying Levi's Lemma [8, see chapter 4, section 11], we can see that there is some d and a differential polynomial $p \in K\{z_1, u_0, u_1, u_2, u_3\}$ such that

- $p \in [z_1]$
- p is homogeneous of degree d in u_1, u_2, u_3 and their derivatives.
- The total degree of p in u_1 is less that d.
- We have the following containment,

$$z_1(u_1^d + p(z_1, u_0, u_1, u_2, u_3)) \in \{G\}. \tag{2}$$

Now substitute $x-\zeta y$ for z_1, $\prod_{\eta \in \mu_5 \backslash \{\zeta\}}(x-\eta y)$ for u_0, $z(\delta y)^2$ for u_1, $-2zy\delta y$ for u_2, and zy^2 for u_3. Now equation 2 becomes:

$$(x-\zeta y)\left(\left(\prod_{\eta \in \mu_5 \backslash \{\zeta\}}(x-\eta y)\right)^d + p(x-\zeta y, \prod_{\eta \in \mu_5 \backslash \{\zeta\}}(x-\eta y), z(\delta y)^2, -2zy\delta y, zy^2)\right) \in \{f\}.$$

As $x-\zeta y \notin [f] : \frac{\partial f}{\partial \delta x}^{\infty}$ it must be that

$$\left(\prod_{\eta\in\mu_5\backslash\{\zeta\}}(x-\eta y)\right)^d + p(x-\zeta y, \prod_{\eta\in\mu_5\backslash\{\zeta\}}(x-\eta y), z(\delta y)^2, -2zy\delta y, zy^2) \in \{f\} \subset [f] : \frac{\partial f}{\partial \delta x}^{\infty}.$$

Recall, $(a,b,0)\in W$, so

$$q(x,y,z) := \left(\prod_{\eta\in\mu_5\backslash\{\zeta\}}(x-\eta y)\right)^d + p((x-\zeta y, \prod_{\eta\in\mu_5\backslash\{\zeta\}}(x-\eta y), z(\delta y)^2, -2zy\delta y, zy^2)$$

vanishes at $(a,b,0)$.

But, $q(a,b,0) = \left(\prod_{\eta\in\mu_5\backslash\{\zeta\}}(a-\eta b)\right)^d$ So, for some $\eta\in\mu_5\backslash\{\zeta\}$,

$$a-\eta b = 0. \tag{3}$$

Combining equations 1 and 3 clearly gives that $a=b=0$. So, the differential algebraic variety W, which is dimension two (see definition 3.1 below) intersects the hyperplane given by $z=0$ which is also dimension two in precisely one point (dimension zero). As both are embeddedd in $\mathbb{A}^3$, the example shows that the above intersection theorem can not hold in the context of differential algebraic geometry.

For an exposition of many of the results used above, see [24]. The exposition here follows the notes of William Sit [23] with several points made by Phyllis Cassidy (via personal correspondence).

The motivating questions of the paper are:

Question 0.2. *Is Ritt's example exceptional or generic for intersections of differential algebraic varieties? More technically: in the space of differential hypersurfaces of a particular order and degree, what is the locus on which the intersection theorem fails for a given arbitrary differential algebraic variety? What is the locus on which the intersection is reducible? Do these loci even have differential algebraic structure?*

The main thrust of the paper is to provide answers to several of these questions (and variants), in part by proving a differential analogue of Bertini's theorem. After this, we point out several applications of the main theorem. In practice, anomalous intersections cause two problems with respect to proving Bertini-style theorems. First, hyperplane sections might not be codimension one (as in the above example), so the dimension conclusion of the theorem will not hold on general grounds once the intersection is proven nonempty (as in the case of algebraic geometry, following from the above affine intersection theorem). Second, the potential for anomalous intersections makes proving irreducibility results more difficult in differential algebraic geometry; the possibility of small dimensional components in intersections are a worry which, again, can be more easily dismissed in the algebraic case.

Dispensing with the first problem for hyperplane sections (or even hypersurface), at least *generically*, is reasonably straightforward. The second problem is slightly more involved. Overcoming it essentially involves several steps: applying differential algebraic reduction theory to establish the primality of a differential ideal over a specific field, using a differential lying over theorem for prime differential ideals, followed by geometric arguments.

Much of the work here is inspired by and related to [28]. The authors of that paper use purely differential algebraic arguements, while we tend to use model theoretic tools as well as the ideas in [28] (appropriately generalized to the partial case). In some cases, model theoretic tools allow proofs to be shortened significantly. Further, we pursue stronger irreducibility results; we are not ultimately interested in irreducibility over a specific differential field. Rather, our goal is to prove *geometric irreducibility* results:

Definition 0.3. An affine differential algebraic variety, V over K, is geometrically irreducible if $I(V/K')$ is a prime differential ideal for *any* K', a differential field extension of K.

In model theoretic terms, geometric irreducibility of V corresponds to the generic type of V over k being stationary. The results [28] are algebraic; in geometric terms, they prove preliminary versions of Bertini-style theorems along the lines of [5, Theorem 2 page 54]. Their results apply to hypersurface sections of affine differential algebraic varieties over a differential field K, cut out by hypersurfaces with coefficients $\bar{u}$, which are new generic indeterminants over K - the results *are not* valid when the coefficients $\bar{u}$ are not independent differential transcendentals (and one considers irreducibility of the intersection over $K\langle\bar{u}\rangle$). However, for applications, one often wishes to take to the coefficients of the hypersurfaces in some specific differential field extension, F, and then consider irreducibility of the intersection over F. Of course, for model theoretic applications, one often wants to assume that K is algebraically or differentially closed, which is never possible using field of the form $K\langle\bar{u}\rangle$ as above. Unfortunately, the irreducibility results of [28] would not be true in that setting even for algebraic varieties without the hypothesis that the dimension of variety considered is greater than one. In fact, we will show that algebraic curves are the only potential counterexamples to geometric irreducibility in generic intersection theorems.

We would also like to point out that some of the results of [28] are connected with (in some cases directly implied by) earlier results of [20] (for instance, both prove results about zero-dimensional differential algebraic varieties avoiding generic hypersurfaces), but since that paper is written from a model-theoretic point of view, the connections may not be obvious to the reader not familiar with the languages of both model theory and differential algebra.

From the model theoretic point of view, most of the interesting questions about *ordinary* differential fields occur at the level of the finite rank types (equivalently, finite transcendence degree differential field extensions). This is essentially because there is precisely one infinite rank type in $S_1(K)$ - namely that of a δ-K-transcendental. Of course, there are interesting questions about the infinite rank definable sets, but these fall outside the interesting lines of work regarding *nonorthogonality*, which is at the heart of geometric stability theory and many model theoretic applications. The situation is similar in partial differential fields, but the line is drawn *not* at the finite-infinite level, but rather at the level of the degree of the Kolchin polynomial being at or below m (the number of derivations). The model theoretic-differential algebraic correspondence is just as strong at this level in the partial case as it is at the finite-infinite level in the ordinary case. Most of the strength of this correspondence is present at other levels of Lascar rank as well (for instance, for $m_1 < m$, considering the types of Lascar rank less than ω^{m_1}). For these intermediate levels of rank, it is not the model theoretic correspondence which breaks down; rather, these differential algebraic varieties, though complicated and in a certain sense infinite dimensional, behave like zero-dimensional algebraic varieties with respect to generic intersections.

After describing the setting and back round in additional detail (section 2), we prove our main intersection theorems over the course of two sections. Most of the differential algebraic arguments are contained in section 3, while the geometric/model-theoretic arguments are mainly confined to section 4. Following this, we concentrate on the smoothness of hyperplane sections in section 5. There are several notions of smoothness for differential algebraic varieties, and we verify that each of the properties is preserved by generic hyperplane sections, which finishes the proof of the main result of the paper:

Theorem 0.4. *($|\Delta| = m$) Let V be a smooth, geometrically irreducible affine differential algebraic variety over a Δ-field K. Let H be a generic hyperplane over K. Then $V \cap H$ is a smooth, geometrically irreducible differential algebraic variety, which is nonempty just in case $dim(V) > 0$. In that case, $V \cap H$ has Kolchin polynomial:*

$$\omega_{V/K}(t) - \binom{t+m}{m}.$$

In section five, we also give several examples and prove several results regarding arc spaces and nonorthogonality (for instance Proposition 5.8).

In the final two sections, we give several applications of the main theorem. In section 6 we give the infinite-dimensional analogue of a model theoretic idea in strongly minimal theories. In a strongly minimal theory Morley rank is definable; take a formula $\phi(\bar{x}, \bar{y})$ over the empty set with parameters $\bar{y}$ from a model $\mathcal{M} \models T$. Then $\{\bar{y} \mid RM(\phi(x, y)) = n\}$ is a definable set. We prove the same fact for differential fields with Δ-transcendence degree playing the role of Morley rank. This allows a more detailed analysis of the exceptional sublocus of the Grassmannian with respect to the

conclusions of 0.4, at least with respect to dimension. Namely, we prove that intersections have appropriate dimension on a Kolchin open subset of the Grassmannian. We do not provide any characterization of the exceptional sublocus with respect to irreducibility. In fact, proving that the exceptional sublocus is a differential algebraic subvariety of the Grassmannian would seem to give a nontrivial reduction to the Ritt problem, as described in the final section.

In section 7, we generally consider the problem of irreducibility in families. Let

$$\phi : V \to S$$

be a morphism of differential algebraic varieties. Is the set

$$\{s \mid \phi^{-1}(s) \text{ is irreducible}\}$$

a constructible set in the Kolchin topology? This problem is equivalent to several well-known and long-standing open problems in differential algebraic geometry, and we do not solve it here. We solve a related problem in the ordinary case. Namely, we answer the question affirmitively when irreducibility is replaced by *generic irreducibility*.

1. Acknowledgements

I would like to gratefully acknowledge the patient and thorough explanations of Ritt's example by William Sit and Phyllis Cassidy. This example, in part, triggered my interest in the problems considered here. Thanks to Phyllis Cassidy for alerting me to the work of Xiao-Shan Gao, Wei Li, and Chun-Ming Yuan. Their techniques had a major influence on the development and exposition of this paper.

This work constitutes a portion of my thesis, supervised by Dave Marker. Thanks to Dave for many useful conversations on the topic. Thanks also to Rahim Moosa for several useful conversations on the problems considered in this paper, several of which took place during a visit to Waterloo sponsored by an American Mathematical Society Mathematics Research Community Grant.

2. Setting and definitions

We will very briefly review some of the developments necessary for our results; more complete expositions can be found in various sources, which we cite below. We use standard model-theoretic notation, following [10] and differential algebraic notation, following [8]. In this paper, we will take K to be a differential field of characteristic zero in m commuting derivations, $\Delta = \{\delta_1, \dots, \delta_m\}$. In this setting, we have a *model companion*, the theory of Δ-closed fields, denoted $DCF_{0,m}$. One may work entirely within a saturated model of $DCF_{0,m}$ for this paper, taking all differential field extensions therein. However, the results of this paper require care with respect to the field we work over. We do not consider abstract differential algebraic varieties or differential schemes; we only consider affine differential algebraic varieties over a

differential field. One can easily extend many of the results of this paper to the projective case, but we do not address this directly.

The *type* (in the sense of model theory) of a finite tuple of $\mathcal{U}$ over F is the collection of all first order formulae with parameters from F which hold of the tuple. A realization of a type p over F (we write $p \in S(F)$) is simply a tuple from a field extension satisfying all of the first order formulae in the type. As $DCF_{0,m}$ has *quantifier elimination*, we have a correspondence between types and prime differential ideals and differential algebraic varieties. Given a type $p \in S(F)$, we have a corresponding (prime) differential ideal via

$$p \mapsto I_p = \{f \in F\{y\} \mid f(y) = 0 \in p\}.$$

The corresponding variety is simply the zero set of I_p. For the reader not acquainted with the language of model theory, the type of a tuple b over F corresponds to the isomorphism type of the differential field extension $F\langle b\rangle/F$ with specified generators. Next, we will give a series of definitions leading to one of the tools used in this paper, Lascar rank, an ordinal valued function on types.

Let Θ be the free commutative monoid generated by Δ. For $\theta \in \Theta$, if $\theta = \delta_1^{\alpha_1} \dots \delta_m^{\alpha_m}$, then $ord(\theta) = \alpha_1 + \dots + \dots + \alpha_m$. The order gives a grading on the monoid Θ. We let

$$\Theta(s) = \{\theta \in \Theta : ord(\theta) \leq s\}$$

Let k be an arbitrary differential field.

Theorem 2.1. *(Theorem 6, page 115, [8]) Let $\eta = (\eta_1, \dots, \eta_n)$ be a finite family of elements in some extension of k. There is a numerical polynomial $\omega_{\eta/k}(s)$ with the following properties.*

1. *For sufficiently large $s \in \mathbb{N}$, the transcendence degree of $k((\theta\eta_j)_{\theta\in\Theta(s),\, 1\leq j\leq n})$ over k is equal to $\omega_{\eta/k}(s)$.*
2. *$deg(\omega_{\eta/k}(s)) \leq m$*
3. *One can write*

$$\omega_{\eta/k}(s) = \sum_{o \leq i \leq m} a_i \binom{s+i}{i}$$

In this case, a_m is the differential transcendence degree *of $k\langle\eta\rangle$ over k.*

Definition 2.2. The polynomial from the theorem is called the *Kolchin polynomial* or the *differential dimension polynomial.* Let $p \in S(k)$. Then $\omega_p(t) := \omega_{b/k}(t)$ where b is any realization of the type p over k.

Definition 2.3. Suppose that p and q are types such that q extends p. This means $p \in S(k)$ for some differential field k and $q \in S(K)$ where K is a differential field extension of k; further, as a sets of first order formulae $p \subset q$.

In this case, any realization of q is necessarily a realization of p when we consider only the formulae which are over k.

Definition 2.4. Let q extend p. We say q is a *nonforking extension* of p if

$$w_p(t) = \omega_q(t).$$

Note that the Kolchin polynomial on the left is being calculated over k and the Kolchin polynomial on the right is being calculated over K.

Definition 2.5. Let p be a type. Then,

- $RU(p) \geq 0$ if p is consistent.
- $RU(p) \geq \beta$, where β is a limit just in case $RU(p) \geq \alpha$ for all $\alpha < \beta$.
- $RU(p) \geq \alpha + 1$ just in case there is a forking extension q of p such that $RU(q) \geq \alpha$.

Remark 2.6. The last three definitions are specific instances of model theoretic notation in the setting of differential algebra; for the more general definitions, see [16]. Our development of this is rather nonstandard; normally, forking is defined in a much more general manner. That forking specializes to the above notion in differential algebra requires proof, but is a natural consequence of the basic model theory of differential fields, see [13].

When considering model theoretic ranks on types like Lascar rank (denoted $RU(p)$) we will write $RU(V)$ for a definable set (whose Δ-closure is irreducible) for $RU(p)$ where p is a type of maximal Lascar rank in V. We should note that some care is required, since the model theoretic ranks are not always invariant under taking Δ-closure. See [2] for an example.

We will be using Lascar rank at various points, and remind the reader of the following result, which we use implicitly throughout the section:

Proposition 2.7. [13] *Let b be a tuple in a differential field extension of k. Then*

$$dim(b/k) = n \text{ if and only if } \omega^m \cdot n \leq RU(tp(b/k)) < \omega^m \cdot (n+1)$$

Definition 2.8. Let $\Delta = \{\delta_1, \ldots, \delta_m\}$. Let $\Delta' = \{\delta'_1, \ldots, \delta'_m\}$. A homomorphism ϕ from Δ-ring (R, Δ) to Δ'-ring (S, Δ') is called a differential homomorphism if for each i, $\phi \circ \delta_i = \delta'_i \circ \phi$. When R is an integral domain and S is a field, then such a map is called a Δ-*specialization.*

Remark 2.9. Differential algebraists sometimes use $\bar{y}$ to denote a specialization of y. Since model theorists often denote tuples of formal variables in this way, we will avoid this notation. For instance, if the specialization is given by a homomorphism ϕ, then when necessary we will write $\phi(y)$ for the specialization of y. The following proposition is proved in a constructive manner in [28, Theorem 2.16], but seems to be a natural consequence of the model-theoretic setup since the positive atomic formulae witnessing non-independence are pushed forward by the differential homomorphism, witnessing non-independence in the image.

Proposition 2.10. *Let $\bar{u} = (u_1, \ldots, u_r) \subset \mathcal{U}$ be a set of Δ-independent differential transcendental elements over K. Let $\bar{y} = (y_1, \ldots, y_n)$ be a set of differential indeterminants. Let $P_i(\bar{u}, \bar{y}) \in K\{\bar{u}, \bar{y}\}$ for $i = 1, \ldots, n_1$. Suppose $\phi : K\{\bar{y}\} \to \mathcal{U}$ be a differential specialization into $\mathcal{U}$ such that $\phi(y_i) \downarrow_K K\langle\bar{u}\rangle$. Suppose that $P_i(\bar{u}, \phi(\bar{y}))$ are (as a collection), not independent over $K\langle\bar{u}\rangle$. Then let ψ be a differential specialization from $K\langle\bar{u}\rangle \to K$. The collection $\{P_i(\psi(\bar{u}), \phi(\bar{y}))\}_{i=1,\ldots,n_1}$ are not independent over K.*

3. Intersections

In this section we develop an intersection theory for differential algebraic varieties with generic Δ-polynomials. The influence of [28] for proving statements about irreducibility over specific differential fields is obvious; we have adapted their techniques to the partial differential setting. Our arguments about dimensions of intersections were done earlier from a more model-theoretic point of view. The following definition matches that of [28, if restricted to the ordinary case].

Definition 3.1. Let X be a Δ-K-variety. Denote, by $dim(X/K)$ the Δ-transcendence degree of a generic point on X over K. As usual, via the correspondence between types, tuples, and differential varieties we will abuse notation and write $dim(p)$ for $p \in S(K)$ or $dim(\bar{a}/K)$ for some tuple in a Δ-field extension.

Definition 3.2. In $\mathbb{A}^n$, the differential hypersurfaces are the zeros of a Δ-polynomial of the form

$$a_0 + \sum a_i m_i$$

where m_i are differential monomials in $\mathcal{F}\{y_1, \ldots, y_n\}$. For convenience, in the following discussion, we do not consider 1 to be a monomial. A *generic Δ-polynomial* of order s and degree r over K is a Δ-polynomial

$$f = a_0 + \sum a_i m_i$$

where m_i ranges over all differential monomials of order less than or equal to s and degree less than or equal to r and $\bar{a} = (a_0, a_1, \ldots, a_n)$ is Δ-transcendental over K. A *generic Δ-hypersurface* of order s and degree r is the zero set of a generic Δ-polynomial of order s and degree r. When f is given as above, we let $\bar{a}_f$ be the tuple of coefficients of f. Throughout, we adopt the notation $\bar{a}_f = a_f \backslash \{a_0\}$.

Remark 3.3. As Example 0.1 shows, not every component of a differential hypersurface would has corresponding ideal which is differentially principal (sometimes one must saturate by the separant). Determining the generators of a differential ideal when it is given in characteristic set form is an open problem [8, see page 170, comments preceding Proposition 5 for details]. In general, one might wish to define differential hypersurfaces as being those varieties whose corresponding ideals have characteristic sets consisting of a single differential polynomial. For purposes of this

paper, the differences are not pertinent, so we can look at the smaller class as defined above.

The next lemma is proved in the ordinary case in [28] (Lemma 3.5). The proof in this case works similarly, assuming that one sets the stage with the proper reduction theory in the partial case. One might notice that Lemma 3.5 of [28] has a second portion. For now, we will concentrate only on the irreducibility of the intersection. Necessary and sufficient conditions for the intersection to be nonempty will be given later.

Lemma 3.4. *Let $\mathfrak{I}$ be a prime Δ-ideal in $K\{\bar{y}\}$ with differential transcendence degree d and let $f = y_0 + \sum_{i=1}^n a_i m_i$ with $(a_1, \ldots, a_n)$ differentially independent over K. Then $\mathfrak{I}_0 = \{\mathfrak{I}, f\}$ is a prime Δ-ideal of $K\langle \bar{a}_f \rangle\{\bar{y}, y_0\}$.*

Proof. Let $\bar{b} = (b_1, \ldots, b_n)$ be a generic point of $V(I)$ over K such that $\bar{b}$ is independent from $\bar{a}$ over K. In model theoretic terms, $\bar{b} \downarrow_K \bar{a}$. Let $f = y_0 + \sum_{i=1}^n a_i m_i$. Consider the tuple $(b_1, \ldots, b_n, -\sum_{i=1}^n a_i m_i(\bar{b}))$. Let $\mathfrak{I}_0 = [\mathfrak{I}, f]$. We show irreducibility of the variety $V(\mathfrak{I}_0)$ in $\mathbb{A}^{n+1}$ via showing that it is the Kolchin closure of $(b_1, \ldots, b_n, -\sum_{i=1}^n a_i m_i(\bar{b}))$ over K. Since only irreducible sets over K have K-generic points in the Kolchin topology, this will complete the proof (said another way, being the locus over K of a tuple in a differential field extension is precisely equivalent to being an irreducible Δ-K-closed set).

Suppose g is a Δ-polynomial over $K\langle \bar{a}_f \rangle$ which vanishes at $(b_1, \ldots, b_n, a_0)$. Consider the f as a Δ-polynomial of $K\langle \bar{a}_H \rangle\{\bar{y}, y_0\}$. Fix a ranking so that y_0 is the leader of f. Then reducing g with respect to f gives some g_0 (which is equivalent to g modulo f). This g_0 must be in $K\langle \bar{a}_f \rangle\{y_1, \ldots, y_n\}$. Of course, since $\bar{b}$ is generic for $\mathfrak{I}$, we must have that $g_0 \in K\langle \bar{a}_f \rangle \cdot \mathfrak{I}$. But then $g \in \mathfrak{I}_0$ and the claim follows. □

Lemma 3.5. *Following the notation of the previous lemma assume $d > 0$; then $RU(V(\mathfrak{I}_0)) = RU(V(\mathfrak{I}))$.*

Proof. Take a nonforking extension of a generic type of $V(I)$ to $K\langle \bar{a}_f \rangle$. Let $\bar{b} = (b_1, \ldots, b_n)$ be a realization of this type. Then $(\bar{b}, -\sum_{i=1}^n a_i m_i(\bar{b}))$ is a point on $V(\mathfrak{I}_0))$. So, $RU(V(\mathfrak{I}_0)) \geq RU(V(\mathfrak{I}))$. On the other hand, for any point $\bar{c} \in V(\mathfrak{I}_0)$, c_{n+1} is in the field $K\langle \bar{a}_f \rangle(c_1, \ldots, c_n)$. In model theoretic terms, the $n+1$st coordinate is necessarily in the definable closure of the first n. $RU(\bar{c}/K\langle \bar{a}_f \rangle) = RU((c_1, \ldots, c_n)/K\langle \bar{a}_f \rangle)$. This establishes $RU(V(\mathfrak{I}_0)) = RU(V(\mathfrak{I}))$. □

Corollary 3.6. *Suppose that the Δ-transcendence degree of $\mathfrak{I}$ is equal to d. Then the Δ-transcendence degree of $\mathfrak{I}_0$ is equal to d.*

Now we turn towards establishing necessary and sufficient conditions for the intersection to be nonempty when we relax the sorts of intersections under consideration. In the case that the intersection is nonempty, we calculate the differential transcendence degree.

Lemma 3.7. *Suppose that V is a differential algebraic variety such that $RU(V/K) < \omega^m$. Then $V \cap V(f(\bar{x})) = \emptyset$ for any generic differential polynomial $f(\bar{x})$.*

Proof. This was originally proved in [21] in the ordinary case, and was reproved in [28] in the ordinary case. The proof for hypersurfaces in the partial case can be found in [1, Proposition 4.1]. □

Lemma 3.8. *Suppose that V is a differential algebraic variety embedded in $\mathbb{A}^n$ with $\omega^m \cdot n_1 \leq RU(V/K) < \omega^m \cdot (n_1 + 1)$ where $n_1 \geq 1$. If $f(\bar{x})$ is a generic differential polynomial, then*

$$\omega^m \cdot (n_1 - 1) \leq RU(V \cap V(f)/K_1) < \omega^m \cdot n_1,$$

where K_1 is the differential field extension of K generated by the coefficients appearing in f.

Proof. Let us prove that the Lascar rank is at least $\omega^m \cdot (n_1 - 1)$. Suppose that for a realization of the generic type, $b_1, \ldots, b_{n_1-1}$ are Δ-dependent mod $\mathfrak{I}_1$, where perhaps we rearrange the coordinates so that $y_1, \ldots, y_{n_1}$ is such that for a generic realization on $\bar{b} \in V(\mathfrak{I})$, $b_1, \ldots, b_{n_1}$ are a Δ-transcendence base for the differential function field generated by $\bar{b}$. Then we get some $f \in K\{\bar{a}_H, b_1, \ldots, b_{n_1-1}, -\sum a_i b_i\}$ and we see that $b_1, \ldots, b_{n_1-1}, -\sum a_i b_i$ are dependent over $K\langle \bar{a}_H \rangle$. Now specialize a_d to -1 and specialize all other $a_i \in \bar{a}_H$ to 0. But then $b_1, \ldots, b_d$ are dependent over K by 2.10, a contradiction.

The upper bound follows easily from considering the projection of the last coordinate of the variety in 3.5 and applying the Lascar inequality. □

Lemma 3.9. *Let $\mathfrak{I}$ be a prime Δ-ideal in $K\{y_1, \ldots, y_n\}$. Let $f = y_0 + \sum_{i=1}^n a_i m_i$ give a generic hypersurface. Then $\mathfrak{I}_1 = \{\mathfrak{I}, f\}$ is a prime Δ-ideal in $K\langle a_0, a_1, \ldots, a_n \rangle \{y_1, \ldots, y_n\}$.*

Proof. In the case that $\tau(V(\mathfrak{I})) < m$, that is, the differential transcendence degree is zero, $V(I) \cap V(f) = \emptyset$ by Lemma 3.7. Thus $\{\mathfrak{I}, f\} = K\langle y_0, a_1, \ldots, a_n \rangle \{y_1, \ldots, y_n\}$.

Now suppose that $\tau(V(I)) = m$, that is, the differential transcendence degree is at least one. Then by Lemma 3.8, $V(I) \cap V(f) \neq \emptyset$. Recall the notation of $\mathfrak{I}_0$ from Lemma 3.4. We will show that $\mathfrak{I}_1 \cap K\langle a_1, \ldots, a_n \rangle \{y_1, \ldots, y_n, y_0\} = \mathfrak{I}_0$. Suppose that we have $g, h \in K\langle a_1, \ldots, a_n, y_0 \rangle \{y_1, \ldots, y_n\}$ such that $g \cdot h \in \mathfrak{I}_1$. Since we are taking a field extension over K, the coefficients of the differential polynomials might involve differential rational functions in $a_1, \ldots, a_n, y_0$ over K. This is easily dispensed with since if we multiply by suitable differential polynomials in $a_0, \ldots, a_n, y_0$ over K, we will get $g, h \in K\{a_1, \ldots, a_n, y_0, y_1, \ldots, y_n\}$ such that $g \cdot h \in \mathfrak{I}_0$. But, $\mathfrak{I}_0$ is prime by Lemma 3.4. So, we have a contradiction and $\mathfrak{I}_1$ is prime. Further, we can see (again, simply by clearing denominators) that $\mathfrak{I}_1$ lies over $\mathfrak{I}_0$, when we regard $\mathfrak{I}_0$ as an ideal of $R\{y_1, \ldots, y_n\}$ where $R = K\langle a_1, \ldots, a_n \rangle \{y_0\}$. □

Proposition 3.10. *Using the notation of the previous lemma and assuming that $dim(V(\mathfrak{I})) = d$ implies that $dim(V(\mathfrak{I}_1)) = d - 1$.*

Proof. Now suppose without loss of generality that $b_1, \dots, b_d$ are independent Δ-transcendentals where $\bar{b}$ is a generic point on $V(\mathfrak{I})$ and $\bar{b} \in V(f)$. We know that $y_0, y_1, \dots, y_d$ are not independent mod $\mathfrak{I}_0$, so we know $b_1, \dots, b_d$ are not independent over $K\langle \bar{a} \rangle$. We claim that $\bar{b}_1, \dots, b_{d-1}$ are independent over $K\langle \bar{a} \rangle$. Suppose not. Then there is some Δ-polynomial $f(x_1, \dots, x_{d-1}) \in \mathfrak{I}_1$. But then there is some Δ-polynomial $\tilde{f}(x_1, \dots, x_{d-1}, y_0) \in \mathfrak{I}_0$, which contradicts lemma 3.4. □

When we assume that the hypersurface is actually a hyperplane, we can make a more detailed calculation:

Lemma 3.11. *Following the notation of the previous lemma, let $d > 0$. Let f be order 0 and degree 1 (that is $V(f)$ is a generic hyperplane). Then,*

$$\omega_{V([\mathfrak{I}, f])/K\langle y_0, a_1, \dots, a_n \rangle}(t) = \omega_{V(\mathfrak{I})/K}(t) - \binom{t+m}{m}$$

Proof. In the notation of 3.9, b_d rational over $b_1, \dots, b_{d-1}$ in $K\langle a_f \rangle$. So, $\omega_{\bar{b}/K\langle a_f \rangle}(t) = \omega_{b_1, \dots, b_{d-1}, d_{d+1}, \dots, b_n/K\langle a_f \rangle}(t)$. Further, by inspecting the differential ideals, we can see

$$\omega_{b_1, \dots, b_{d-1}, d_{d+1}, \dots, b_n/K\langle a_f \rangle}(t) = \omega_{b_1, \dots, b_{d-1}, d_{d+1}, \dots, b_n/K}(t) = \omega_{\bar{b}/K}(t) - \binom{t+m}{m}$$

□

Putting together results 3.4, 3.5, 3.7, 3.8, 3.9, and 3.11 we have proved the following theorem,

Theorem 3.12. *Let V be a Kolchin-closed (over K) subset of $\mathbb{A}^n$ with differential transcendence degree d. Let H be a generic (with respect to K) hypersurface corresponding to the tuple $\bar{a}$ (as above). Then $V \cap H$ is a Kolchin-closed subset of $\mathbb{A}^n$ with differential transcendence degree $d-1$. $V \cap H$ is irreducible over $K\langle \bar{a} \rangle$. In the case that $d = 0$, $V \cap H = \emptyset$. If $d > 0$ and H is a generic hyperplane, then the Kolchin polynomial of $V \cap H$ is given by*

$$\omega_{V \cap H/K\langle a_H \rangle}(t) = \omega_{V/K}(t) - \binom{t+m}{m}$$

Remark 3.13. Note that we are considering the Kolchin topology over a *specific field*, and not its differential (or even algebraic) closure. Irreducibility over the algebraic closure of the differential field of definition is what we call geometric irreducibility. We have not proved this yet, nor do the authors of [28] in the ordinary setting. In fact, at least one additional hypothesis is necessary for that result: if the hypothesis were *purely in terms of dimension*, we would have to restrict to the situation $d \geq 2$. After all, take any degree $d_1 > 1$ plane curve. This curve meets the generic hyperplane in precisely d_1 points, so the intersection is not irreducible over any algebraically closed field. In fact, in the next section, we show that this is the only potential problem by

proving a more detailed result which applies to any differential algebraic variety for which the intersection with a generic hyperplane is infinite.

Also note that we have not computed the Kolchin polynomial of the intersection except in the special case of a hyperplane.

4. Geometric irreducibility

Before discussing geometric irreducibility, we will require some results about the Kolchin polynomials of prime differential ideals lying over a fixed prime differential ideal in extensions.

Proposition 4.1. *([8] pg131, proposition 3, part b) Let $\mathfrak{p}$ be a prime differential ideal in $K\{y_1, \dots, y_n\}$ and let F be a differential field extension of K. Then $F\mathfrak{p}$ has finitely many prime components in $F\{y_1, \dots, y_n\}$. If $\mathfrak{q}$ is any of the components, then a generic type of the variety $V(\mathfrak{q})$ has the same Kolchin polynomial as the generic type of $V(\mathfrak{p})$.*

Remark 4.2. In model theoretic terms, the generic types of the components $V(\mathfrak{p}_1), \dots, V(\mathfrak{p}_n)$ of $V(\mathfrak{p})$ are each nonforking extensions of the generic type of $V(\mathfrak{p})$. Assuming that the base field K is algebraically closed would ensure that the generic type of $V(\mathfrak{p})$ is stationary; consequently $F\mathfrak{p}$ is a prime differential ideal for any field extension F of K.

Recall the following definition given in the introduction:

Definition 4.3. An affine differential algebraic variety, V over K, is geometrically irreducible if $I(V/K')$ is a prime differential ideal for *any* K', a differential field extension of K.

Remark 4.4. The previous proposition and remark show that it is enough to consider irreducibility over K^{alg}, the algebraic closure of K. To put geometric irreducibility in the language of differential schemes, if $V = \Delta Spec(K\{x_1, \dots, x_n\}/\mathfrak{p})$ where $\mathfrak{p}$ is a prime differential ideal, then V is geometrically irreducible if its base change $\Delta Spec(K\{x_1, \dots, x_n\}/\mathfrak{p}) \times_{\Delta Spec(K)} \Delta Spec(K^{alg})$ is irreducible.

Theorem 4.5. *Let V be a (geometrically) irreducible Kolchin-closed (over K) subset of $\mathbb{A}^n$ with Kolchin polynomial $\omega_V(t) > \binom{t+m}{m}$. Let H be a generic hyperplane. Then $V \cap H$ is geometrically irreducible and $\omega_{H\cap V}(t) = \omega_V(t) - \binom{t+m}{m}$.*

Remark 4.6. The strict inequality $\omega_V(t) > \binom{t+m}{m}$ in the hypothesis is necessary. This hypothesis prevents V from being an algebraic curve (in fact, it is equivalent to this). The fact we get geometric irreducibility of the intersection even in the case that V is dimension one (as long as V is not an algebraic curve) is in contrast to the case of algebraic geometry.

Proof. Consider the the differential algebraic variety $W = \{(v_1, v_2, \beta) \,|\, v_i \in V,\, v_i \in H_\beta\} \subseteq V \times V \times \mathbb{A}^n$ where H_β is the hyperplane given by $\sum \beta_i x_i = 1$. It might be the case that W is reducible in the Kolchin topology, but we will not be concerned with this specifically.

Consider $V \cap H_\beta$. When β is generic over K, we know that $V \cap H_\beta$ is irreducible over $K\langle\beta\rangle$, so by the Proposition 4.1, all of the components of V over the algebraic closure of K have Kolchin polynomial equal to $\omega_{V \cap H_\beta / K\langle\beta\rangle}(t)$. If $V \cap H_\beta$ has more than one component, then W has more than one component with Kolchin polynomial $2\omega_{V/K} + (n-2)\binom{t+m}{m}$. To see this, consider the complete types given by independent generic points on $V \cap H_\beta$, and β generic subject to the requirement that $v_i \in H_\beta$. Then there is more than one option for the type (v_1, v_2, β), depending on if v_1 and v_2 are in the same component of $V \cap H_\beta$ over $K\langle\beta\rangle^{alg}$. Now, we only consider components of W with Kolchin polynomial $2\omega_{V/K} + (n-2)\binom{t+m}{m}$ and show that $V \cap H_\beta$ is irreducible over $K\langle\beta\rangle^{alg}$.

Suppose v_1 and v_2 are independent generic points on V, and β is generic subject to the condition that H_β contains v_1, v_2. Then the triple v_1, v_2, β is generic on W over K. We will show that this is the only way to construct a generic type on W. If $v_1 \neq v_2$ and β is generic over v_1, v_2, subject to $v_i \in H_\beta$, then $\omega_{\beta/v_1v_2}(t) = \binom{t+m}{m} \cdot (n-2)$. If $v_1 = v_2$, then $\omega_{\beta/v_1}(t) = \binom{t+m}{m} \cdot (n-1)$. But, if $v_1 = v_2$, then $\omega_{v_1v_2\beta}(t) < 2\omega_V + \binom{t+m}{m} \cdot (n-2)$, because $\omega_{v_1/K} > \binom{t+m}{m}$. So, there is a unique type in W with Kolchin polynomial $2\omega_{V/K} + (n-2)\binom{t+m}{m}$, so there is only one component with Kolchin polynomial $2\omega_{V/K} + (n-2)\binom{t+m}{m}$.

By our earlier arguments, there is a unique component of $V \cap H_\beta$ with Kolchin polynomial $\omega_V - \binom{t+m}{m}$. But, by Proposition 4.1, we know any of component of $V \cap H_\beta$ must have Kolchin polynomial $\omega_V - \binom{t+m}{m}$. So, $V \cap H_\beta$ is geometrically irreducible. $\square$

Notice that the main intersection theory results of the last section applied to subvarieties cut out by generic differential polynomials, not just generic hyperplane sections. The exception is the calculation of the Kolchin polynomial of $V \cap H$ in Theorem 3.12. In order to replicate the methods used in this section for arbitrary generic differential hypersurfaces, one would have to provide calculations of the Kolchin polynomial of the intersection of V with arbitrary generic hypersurfaces; this might be possible, but we have not done it here. In the ordinary case, the calculations of the Kolchin polynomial were carried out [28, Theorem 1.1].

5. Smoothness and arc spaces

5.1. **Arc spaces.** Before discussing the smoothness of generic intersections, we will briefly review the construction of differential arc spaces, following [14] for the first part, and [9] thereafter. *None of the results in this subsection are new*; see [14, 15, 9] for complete references. Throughout this subsection: S is a scheme and $T \to S$ is a

scheme over S. Given a scheme Y over T, we let

$$\mathcal{R}_{T/S}Y : \{\text{Schemes over } S\} \to \{\text{Sets}\}$$

be the functor given by

$$\mathcal{R}_{T/S}(U) = Hom_T(U \times_S T, Y)$$

In some situations, $\mathcal{R}_{T/S}$ is a representable functor. When this is the case, we will let $\mathcal{R}_{T/S}(Y)$ be the representing object, that is, the scheme over S. For our purposes, this functor is always representable because we only work in the affine case [15, for more complete details regarding representability].

Example 5.1. Here is a concrete (and pertinent) example of Weil restriction. Let K' be a finitely generated field extension of K and let X' be an affine scheme of finite type over K'. Then the *Weil restriction* $\mathcal{R}_{K'/K}X'(A) = X'(A \otimes_K K')$ for any finitely generated commutative K-algebra A. So,

$$\mathcal{R}_{K'/K}(-) : \{\text{affine K'-schemes}\} \to \{\text{affine K-schemes}\}$$

is right adjoint to base change from K to K'. Fix X' an affine K'-scheme with coordinate ring $K'[X']$ given by $K'[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Then the coordinate ring of the affine K-scheme given by the Weil restriction, $\mathcal{R}_{K'/K}X'$ is given as follows. Let $y_0, y_1, \ldots, y_d$ be a basis for K' over K as a vector space (let us assume $y_0 = 1$). Then

$$K[\mathcal{R}_{K'/K}X'] = K[x_{1,1}, \ldots, x_{1,d}, \ldots x_{n,1}, \ldots, x_{n,d}]/(f_{1,1}, \ldots, f_{m,d})$$

where $f_{j,k} \in K[x_{1,1}, \ldots, x_{1,d}, \ldots, x_{n,1}, \ldots, x_{n,d}]$ are such that:

$$f_j\left(\left(\sum_{j=0}^{m} x_{i,k}y_k\right)\right) = \sum_{k=0}^{m} f_{j,k}y_k$$

Note that this uniquely defines $f_{j,k}$.

Next, consider the case in which again, $S = Spec(K)$. Now let $T = Spec(K^{(m)})$ where

$$K^{(m)} := K[\epsilon]/(\epsilon^{m+1})$$

Let $Y = X \times_S T$, where X is an affine scheme over K. $K^{(m)}$ is a K-algebra with the natural map,

$$a \to a + 0\epsilon + \ldots + 0\epsilon^m.$$

The m^{th} *arc bundle of* X over K is $R_{K^{(m)}/K}(X \times_K K^{(m)})$, the scheme representing the Weil restriction of $X \times_S T$ from T to S. Throughout, we will denote this particular Weil restriction as $\mathcal{A}_m(X/K)$ or $\mathcal{A}_m(X)$ when K is implicit. Of course, for any K-algebra R, $\mathcal{A}_mX(R)$ can be naturally identified with $X(R[\epsilon]/(\epsilon^{m+1}))$.

Now, recalling the remarks in example 5.1, take $X \subseteq \mathbb{A}^l$. Suppose

$$X := Spec(K[x_1, \ldots, x_l]/(\{f_j\}_{j\in J}\})$$

Then

$$\mathcal{A}_m(X) = Spec(K[\{x_{i,s}\}_{1,\leq i\leq l, 0\leq s\leq m}]/(\{f_{j,t}\}_{0\leq t\leq m})$$

where $f_{j,t} \in K[\{x_{i,s}\}_{1,\leq i\leq l,0\leq s\leq m}]$ is defined by,

$$f_j((\sum_{i=0}^{m} x_{i,t}\epsilon^t)_{1\leq i\leq l}) = \sum_{t=0}^{m} f_{j,t}\epsilon^t,$$

calculated in the ring $K[\{x_{i,s}\}, \epsilon]/(\epsilon^{m+1})$.

Now, suppose $f : X \to Y$ is a map of K-varieties. It is a fact that f induces a map on the arc spaces denoted $\mathcal{A}_m(f) : \mathcal{A}_m(X) \to \mathcal{A}_m(Y)$. Suppose $X \subseteq \mathbb{A}^l, Y \subseteq \mathbb{A}^r$. Let us consider this map in more detail. Say $f = (f_1, \ldots, f_r)$. Then we can compute the map $\mathcal{A}_m(f)$ by considering $\mathcal{A}_m(X)$ as points in $\mathbb{A}^l(K[\epsilon]/(\epsilon^{m+1}))$. Then to compute the image of some point $b \in \mathbb{A}^l(K[\epsilon]/(\epsilon^{m+1}))$, one simply computes $f_i(b) \in K[\epsilon]/(\epsilon^{m+1})$. Further, we remark that there is a natural map of arc spaces $\rho_{l,m} : \mathcal{A}_l \to \mathcal{A}_m$ for $l > m$ which is induced by the quotient map on the ring $K[\epsilon]/(\epsilon^l)$. When we refer to m^{th} arc space at a point $a \in X$, we mean the fiber of the map $\rho_{m,0}$.

We will now review the construction of Δ-arc spaces for affine Δ-varieties; again, we follow [14], where complete details are given. Now we assume that K is a Δ-field. Let

$$K_m := K[\eta_1, \ldots, \eta_n]/(\eta_1, \ldots, \eta_n)^{m+1}.$$

We make this ring a K-algebra via the map

$$a \mapsto \sum_{0\leq \alpha_1+,\ldots,+\alpha_n\leq m} \frac{1}{\alpha_1! \cdot \ldots \cdot \alpha_n!} \delta_1^{\alpha_1} \ldots \delta_n^{\alpha_n}(a)\eta_1^{\alpha_1} \ldots \eta_n^{\alpha_n}.$$

Now let $S = spec(K)$, $T = spec(K_m)$. For an algebraic variety, the *m^{th} prolongation $\tau_m X$ of X* is the Weil restriction of $X \times_S T$ from $Spec(K_m)$ to $Spec(K)$.

Note that if $m = 1$ and δ_1 is the trivial derivation, τ_m is the same as $\mathcal{A}_m$. As in the case of $K^{(m)}$, we have reduction maps (given by quotients) from $K_l \to K_m$, and $\pi_{l,m} : \tau_l \to \tau_m$. We denote $\pi_{l,0}$ by π_l. Let $\nabla_m : X \to \tau_m X$ be given by

$$x \mapsto \sum_{0\leq \alpha_1+\ldots+\alpha_n\leq m} \frac{1}{\alpha_1! \cdot \ldots \cdot \alpha_n!} \delta_1^{\alpha_1} \ldots \delta_n^{\alpha_n}(x)\eta_1^{\alpha_1} \ldots \eta_n^{\alpha_n}.$$

Then ∇_m is a Δ-regular section of π_m.

Recall, we only consider affine Δ-varieties, so, throughout, we let X be a Δ-variety over K and $\bar{X}$ be the Zariski closure of X over K. Now we define $\tau_m X$ as the *Zariski closure* of $\nabla_m X(K)$ as a subvariety of $\tau_m \bar{X}(K)$. Note that X is determined completely by the sequence

$$\langle \pi_{l,m} : \tau_l X \to \tau_m X \mid l > m \rangle.$$

Note that

$$X(K) = \{a \in \tau_0 X(K) \,:\, \nabla_l(a) \in \tau_l X(K), \forall l > 0\}$$

On the other hand, define

$$\langle X_l \subseteq \tau_l \bar{X} \mid l \geq 0 \rangle,$$

a sequence of irreducible algebraic subvarieties to be a *prolongation sequence* if

- $\pi_{l+1,l}$ is a dominant map from X_{l+1} to X_l
- Considering $\tau_{l+1}\bar{X}$ as a subvariety of $\tau^{l+1}\bar{X}$, X_{l+1} is a closed subvariety of τX_l

In this case, there is a unique Δ-subvariety X of $\bar{X}$ so that $\tau_l X = X_l$. So, there is a natural equivalence of categories between affine Δ-varieties modulo a proper closed differential algebraic subvariety and prolongation sequences.

Proposition 5.2. [14] *Let $K \models DCF_{0,m}$ and let $X \subseteq \bar{X}$ be irreducible Δ-subvariety of $\bar{X}$, an algebraic subvariety both defined over K.*

$$\langle \mathcal{A}_m(\pi_{s,t}) : \mathcal{A}_m \tau_s X \to \mathcal{A}_m \tau_t X \mid s \geq t \rangle$$

is the prolongation sequence for a Δ-subvariety of $\mathcal{A}_m \bar{X}$.

Definition 5.3. The m^{th} arc bundle of X is $\mathcal{A}_m X$ is the Δ-subvariety specified by the prolongation sequence from the previous proposition.

$\mathcal{A}_1 X$ is naturally isomorphic to the Δ-tangent bundle, which is described below, and for which we use Kolchin's notation $T^{\Delta}X$.

In [14], the authors give the following definition:

Definition 5.4. $a \in X(K)$ is a *smooth point* if $\nabla_s(a)$ is a smooth point on $\tau_s X(K)$ for each s and $d(\pi_{s,t})_{\nabla_s(a)}$ has full rank for every $s \geq t$. When $a \in X(K)$ is smooth, $\tau_s(\mathcal{A}_m X_a) = \mathcal{A}_m(\tau_s X)_{\nabla_s(a)}$.

The remainder of the section follows Kolchin's development of the differential tangent space [9]. For this section, let $X = V(\mathfrak{p})$, a prime differential ideal in the ring $K\{y_1, \ldots, y_n\}$. Let $R = K\{y\}/\mathfrak{p}$. Then for $x \in X(K)$, let $\mathcal{O}_{x,X}$ be the localization of R at the maximal ideal of differential polynomials which vanish at x.

Definition 5.5. A local Δ-derivation on $\mathcal{O}_{X,x}$ is a derivation from $\mathcal{O}_{X,x}$ to K which commutes with the elements of Δ. The set of all Δ-derivations on $\mathcal{O}_{X,x}$ is a vector space over the constants. This space is denoted by $T^{\Delta}_x(X)$.

We have the following natural map:

$$d_x : T^{\Delta}_x(X) \to \mathbb{G}_a^n$$

where $T \mapsto (T\bar{y}_1, \ldots, T\bar{y}_n)$ where $\bar{y}_i$ is the image of y_i in the quotient map $\mathcal{O}_{\mathbb{A}^n,x}/\mathfrak{p}\mathcal{O}_{\mathbb{A}^n,x}$. For $f \in K\{y\}$, define

$$D(f) := \sum_{\theta \in \Theta, i=1,\ldots,n} \frac{\partial f}{\partial \theta y_i} \theta y_i$$

and

$$D(f)_x := \sum_{\theta \in \Theta, i=1,\ldots,n} \frac{\partial f}{\partial \theta y_i}(x) \theta y_i.$$

Then let $\mathfrak{p}_x := [\{D(f)_x \mid f \in \mathfrak{p}\}]$. $T_x^{\Delta}(X)$ is isomorphic to the subgroup of $\mathbb{G}_a^n$ defined by $\mathfrak{p}_x$. So, given ideal of a differential algebraic variety, it is easy to construct the ideal of the differential tangent space at a point.

This philosophy *almost* works for characteristic sets. In order to ensure that the characteristic set of the ideal of the differential tangent space at a point $\bar{a}$ has the same leaders as the characteristic set of the ideal of the original variety, it is necessary to assume that the initials and separants of a characteristic set of the variety do not vanish at $\bar{a}$. If one only considers orderly rankings, this leads to an easy proof that at points where the initials and separants do not vanish, the differential tangent space and the original variety have the same Kolchin polynomial. For complete details, see [25].

But while the conditions described in the above paragraph are sufficient for the differential tangent space to have the same Kolchin polynomial; they are certainly not necessary. One might take the definition of smooth to depend on the nonvanishing of initials and separants under a given orderly ranking, but this approach has the unwanted property of depending on the ranking chosen. Perhaps it would be interesting to quantify over all orderly rankings for the definition of smoothness, but we do not pursue this line here.

Now, we give a second very natural definition of smoothness for differential algebraic varieties:

Definition 5.6. A point $a \in V$ is smooth if $T_a^{\Delta}V$ has the same Kolchin polynomial as V, that is:

$$\omega_{V/K}(t) = \omega_{T_a^{\Delta}V/K\langle a\rangle}(t)$$

Remark 5.7. This definition of smoothness is not equivalent to the earlier one given 5.4.

5.2. **Inheriting orthogonality and regularity from Δ-tangent spaces.** In this subsection we take a short detour and consider the model theoretic relation of orthogonality and its relationship with arc spaces. Of course, work along these lines was undertaken in several previous works [14] [18], where arc spaces were used to prove partial dichotomy-style theorems dividing differential algebraic varieties into two nonorthogonality classes: those whose geometries are complicated, but whose nonorthogonality classes have varieties with defining ideals that are linear and those whose geometries are rather simple. We will not pursue such goals in this subsection, but we wish to show a few applications of arc spaces, especially in light of the work [25] regarding the heat equation.

Recall that for two types p, q, we say p is *orthogonal* to q and write $p \perp q$ if for all A containing the domains of p and q, and realizations $a \models p|_C, b \models q|_C$ of the nonforking extensions, we have $a \mathop{\smile\!\!\!\!\!\!|}_C b$. A type is *regular* if it is orthogonal to any forking extension. In this subsection we will pursue the following idea: nonorthogonality of

two Kolchin-closed sets should induce nonorthogonality of their differential tangent spaces.

Suppose that $X_1 \not\perp X_2$. Then, over some sufficiently large parameter set, A, we can find $a_1 \in X_1$ and $a_2 \in X_2$ such that $Z = V(I(a_1, a_2))$ is a proper Kolchin-closed subset of $X_1 \times X_2$, and the projections $\pi_1 : Z \to X_1$, $\pi_2 Z \to X_2$ are Kolchin dense.

So, there is a natural inclusion $T^\Delta Z$ in $T^\Delta(X_1 \times X_2)$. Now, suppose we could find $c \in Z$ so that $\pi_1(c)$ and $\pi_2(c)$ are generic in $X_1 \times X_2$. Further, demand that $\nabla(\pi_i(c)) \in T^\Delta X_i$ is of full rank (this is an open condition on X_i, certainly, this is implied by taking $\pi_i(c)$ to be smooth in the sense of definition 5.4).

Then $\nabla(\pi_1(c)), \nabla(\pi_2(c))$ are generic in $T^\Delta_{\pi_i(c)}(X_i)$, respectively. But, the Kolchin polynomial of the pair $(\nabla(\pi_1(c)), \nabla(\pi_2(c))) \in T^\Delta_{(\pi_1(c),\pi_2(c))}(X_1 \times X_2)$ is bounded by the Kolchin polynomial of a generic point in $T^\Delta_c(Z)$. Assuming c is a smooth point (see 5.4 - again, this is an open condition) on Z, the Kolchin polynomial for our pair is bounded by the Kolchin polynomial for a generic point in Z. Of course, this is strictly less than the Kolchin polynomial for a generic point on $X_1 \times X_2$, which, at least at smooth points, has the same Kolchin polynomial as $T^\Delta_{(a_1,a_2)}(X_1 \times X_2)$. Thus, it must be the case that $V(I((\nabla(\pi_1(c)), \nabla(\pi_2(c)))))$ is a proper subvariety of $T^\Delta_{(\pi_1(c),\pi_2(c))}(X_1 \times X_2)$. So, $\nabla(\pi_1(c)) \not\perp_A \nabla(\pi_2(c))$. This means that as definable sets, $T^\Delta_{\pi_1(c)}X_1 \not\perp T^\Delta_{\pi_2(c)}X_2$.

So, we have found a sufficient condition for orthogonality of two types based on orthogonality of their Δ-tangent spaces above sufficiently general points:

Proposition 5.8. *Suppose that the generic types of $T^\Delta_{x_1}X_1$ and $T^\Delta_{x_2}X_2$ are orthogonal where $x_i \in X_i$ generic. Then the generic types of X_1 and X_2 are orthogonal.*

Remark 5.9. This condition does not manifest in a meaningful way for ordinary differential fields, because in that setting $T^\Delta_{\pi_1(c)}X_1 \not\perp T^\Delta_{\pi_2(c)}X_2$ for any finite rank X_1 and X_2. In that setting, for finite rank differential algebraic varieties, differential tangent spaces (like all linear differential algebraic varieties) are finite dimensional vector spaces over the constants. Further, as we noted in the introduction, nonorthogonality is not a meaningful dividing line for positive dimensional differential algebraic varieties. However, in partial differential algebraic geometry, there is a greater diversity of types modulo nonorthogonality, assuming we allow types with nonconstant Kolchin polynomials.

We will give several examples later in this section. First, we prove another corollary which is completely trivial for ordinary differential algebraic geometry.

Corollary 5.10. *Let V be a differential algebraic variety over K. Let $a \in V$ be generic over K. Let $c \in T^\Delta_a V$ be generic over $K\langle a\rangle$. If $tp(c/K\langle a\rangle)$ is regular then $tp(a/K)$ is regular. Also,*

$$RU(a/K) \leq RU(c/K\langle a\rangle)$$

Proof. If $tp(a/K)$ is nonorthogonal to some forking extension, then this induces nonorthogonality of $tp(c/K)$ to some forking extension - namely, the generic type of differential tangent space (at a generic point) of the locus of a realization of the forking extension of $tp(a/K)$. ☐

Example 5.11. Let $K_0 \models DCF_{0,\Delta}$ with $\Delta = \{\delta_1, \delta_2\}$. We will study the differential variety, X, defined by the equation:

$$(\delta_1^2 x)^2 = (\delta_2^3 x)^3.$$

Of course, the Δ-tangent space of this variety at a smooth point is isomorphic to the Heat equation, which, by [26], has Lascar rank equal to ω. Throughout this discussion E will be the curve $y^2 = x^3$.

Let $c \in \mathcal{U}$ be a δ_2 transcendental over K_0. Further, let $d^2 = c^3$. Let $b \in \mathcal{U}$ be an element such that $b \models \delta_2^3(x) = \delta_1(c) \wedge \delta_1^2(x) = d$. In fact, we assume that the positive type of b in $S_1(K_0\{c, d\})$ is isolated by the given formulas (equivalently, b is a generic point on the given Δ-variety, in the Kolchin topology).

Now, take $e \in \mathcal{U}$ to be a generic solution to the equation $\delta_2^3(x) = c$. We let $F = K_0\{b, c, d\}$ and we consider

$$I_F^e(\delta_1) := \{T : F\{e\} \to F\{e\} \,|\, T \text{ is a } \Delta\text{-derivation and } T|_F = \delta_1\}.$$

From work of Kolchin ([8] chapter 2), we know $I_F^e(\delta_1)$ actually has the form of a $\Delta-$variety, $\mathcal{G}(\delta_1)$. In this case, since the positive type of e over F is implied by $\delta_2^3 x = c$, we have that $\mathcal{G}(\delta_1)$ is the zero set of the differential ideal generated by:

$$\mathfrak{p}^e_{F,\delta_1} := \{\delta_2^3 x - \delta_1(c)\}.$$

So, there is $T \in I_F^e(\delta_1)$ with $T(e) = b$ if and only if $b \in \mathcal{G}(\delta_1)$. But, indeed, by the choice of b, we know that $b \in \mathcal{G}(\delta_1)$. So, there is some $T \in I_F^e(\delta_1)$ with $T(e) = b$. Then, $K_0 \models DCF_{0,\Delta'}$ where $\Delta' = \{T, \delta_1\}$. But, this means that c is a δ_2-transcendental over K_0 and $c \models (T^2 x)^2 = (\delta_2^3 x)^3$. One can easily write down enough forking extensions of $tp(c/K_0)$ to see that $RU(tp(c/K_0)) \geq \omega$. By work of Suer [25], the Lascar rank of the heat equation is ω, and by corollary 5.10, we know that the Lascar rank of the (generic point of the) variety is bounded by the Lascar rank of its differential tangent space. So, $RU(tp(c/K_0)) = \omega$.

Similar analysis of other varieties whose generic differential tangent spaces are subgroups of the additive group which are rank ω easily lead to other examples of regular types in differential fields. For other subgroups of the additive group of rank ω, see [25, Proposition 3.45, for instance].

5.3. **Smoothness and hyperplane sections.** We now return to our analysis of hyperplane sections of differential algebraic varieties.

Question 5.12. *Suppose that a differential algebraic variety V is smooth. Then, for sufficiently general hyperplane H, is $V \cap H$ smooth?*

First, we will answer this question affirmatively for definition 5.6.

Proposition 5.13. *Let V be an irreducible smooth Kolchin-closed (over K) subset of $\mathbb{A}^n$ 5.6. Let H be a generic (with respect to K) hypersurface. Then $V \cap H$ is smooth.*

Proof. In the following proof, we are using a technique similar to that of lemmas 3.4 through 3.11. We will be using the calculations of Kolchin polynomials of those results, as well as the irreducibility results in those lemmas.

Consider, as a Kolchin-closed subset of $\mathbb{A}^{n+1}$, the locus of $f = y_0 + \sum_{i=1}^n a_i y_1$ and $I(V/K\langle a_1, \dots, a_n\rangle)$. We will call this differential algebraic variety W. We note that by Lemma 3.4, W is irreducible. Let $\bar{b} = (b_0, \dots, b_n) \in W$. Then consider $T_{\bar{b}}^{\Delta} W$.

Let $\hat{b} = (b_1, \dots, b_n)$. Then fix some orderly ranking $<_1$ on $y_1, \dots, y_n$ and let $\not\geq_{\hat{b}}$ be a characteristic set of the differential tangent space $T_{\hat{b}}^{\Delta} V$. We note that by [8] Theorem 6, section 2.6, the Kolchin polynomial of a variety is determined by the leaders of its characteristic set with respect to any orderly ranking. But, now consider orderly ranking $<_2$ for which $y_i <_2 y_0$ for all $i > 0$ and for all differential monomials $m_1, m_2 \in \Theta(y_1, \dots, y_n)$, $m_1 <_1 m_2$ if and only if $m_1 <_2 m_2$. Then $\Lambda_{\hat{b}}$ together with f is a characteristic set for $T_{\bar{b}}^{\Delta} W$. To see this, note that the set is autoreduced and coherent with respect to $<_2$. $\Lambda_{\hat{b}}$ is a characteristic set under $<_1$, and no derivative of the leader of f appears in $\Lambda_{\hat{b}}$.

We note that by [8] Theorem 6, section 2.6, the Kolchin polynomial of a variety is determined by the leaders of its characteristic set with respect to any orderly ranking. Then

$$\omega_{T_{\bar{b}}^{\Delta} W/K\langle \bar{b}\rangle}(t) = \omega_{T_{\hat{b}}^{\Delta} V/K\langle \hat{b}\rangle}(t)$$

Now, let H be the hyperplane in $\mathbb{A}^n$ given by f over $K\langle \bar{a}, y_0\rangle$. It is obvious that

$$\omega_{T_{hatb}^{\Delta}(V\cap H)/K\langle \bar{a}, \hat{b}\rangle}(t) = \omega_{T_{\bar{b}}^{\Delta} W/K\langle \bar{b}\rangle}(t) - \binom{t+m}{m}$$

We can also see that

$$\omega_{V\cap H/K\langle \bar{a}, y_0\rangle}(t) = \omega_{W/K\langle \bar{a}\rangle}(t) - \binom{t+m}{m} = \omega_{V/K}(t) - \binom{t+m}{m}$$

So,

$$\omega_{V\cap H/K\langle \bar{a}, y_0\rangle}(t) = \omega_{T_{\hat{b}}^{\Delta}(V\cap H)/K\langle \bar{a}, \hat{b}\rangle}(t)$$

follows from the smoothness assumption on V and the above equations. □

Now, we will answer question 5.12 affirmitively for definition 5.4. Assume $V \subset \mathbb{A}^n$. Here, we can apply [4, Corollary 10.9 without the assumption of projectivity, see remark 10.9.2 following that corollary] because for each l, the collection $\{\tau_l H\}$ where H ranges over the hyperplanes of $\mathbb{A}^n$ forms a linear system in $\tau_l \mathbb{A}^n$. For all l, $\tau_l V$ is a variety over the field of definition of V as a *differential algebraic variety*, and we are taking H generic over this field. Then $(\tau_l V \backslash (\tau_l V)_{sing}) \cap \tau_l H$ is smooth. Further, by

the hypotheses of the smoothness of V, for any $a \in V$, $\nabla_l(\bar{a}) \in \tau_l V \backslash (\tau_l V)_{sing}$, so for $a \in V \cap H$, $\nabla_l(\bar{a}) \in \tau_l V \backslash (\tau_l V)_{sing} \cap \tau_l H$.

We must verify that $d(\pi_{s,t})_{\nabla_s(a)}$ has full rank for every $s \geq t$. Fix $a \in V$ generic and independent from $H = H_{\bar{b}}$ (in the notation of section 3). Then $\tau_{s+1} H_{\nabla_s(a)}$ is a generic hyperplane (with respect to the differential field generated by the canonical parameter of X and a) in $\tau_{s+1} V_{\nabla_s(a)}$. So, as long as $(X_{s+1})_{\nabla_s(a)}$ is not zero dimensional $(X_{s+1} \cap \tau_{s+1} H)_{\nabla_s(a)}$ is nonempty. However, assume that V is not zero dimensional *as a differential algebraic variety* so that $V \cap H$ is nonempty. Then the closed subvariety $W \subset \bar{V}$ (the Zariski closure of V) such that for $a \in W$, $(X_{s+1})_{\nabla_s(a)}$ is positive dimensional must be nonempty. On the closed subvariety W, the map $d(\pi_{s+1,s})_{\nabla_s(a)}$ has full rank. If $V \backslash W$ is nonempty, then as a constructible set in the Kolchin topology, it is zero-dimensional. Thus $V \backslash W \cap H$ is empty by Proposition 3.7. Thus, we have established:

Theorem 5.14. *Let V be an irreducible smooth (5.4 or 5.6) Kolchin-closed (over K) subset of $\mathbb{A}^n$. Let H be a generic (with respect to K) hypersurface. Then $V \cap H$ is smooth.*

6. Definability of rank

We next turn to a couple of applications of geometric or model theoretic nature. First, we will prove differential transcendence degree is a constructible condition in the Kolchin topology:

Lemma 6.1. *Given a family of differential algebraic quasi-varieties, $\phi : X \to S$, with $a_m(S) = 0$, the set $\{s \in S \mid a_m(X_s) = d\}$ is a constructible subset of S.*

Proof. Fix d $n+1$-tuples $(c_{i,j})_{1 \leq i \leq d, 1 \leq j \leq n+1}$ such that the elements in the tuple are independent Δ-transcendentals over the canonical bases of X, S, and ϕ. Then by Theorem 3.12,

$$\{s \mid a_m(\phi^{-1}(s)) \geq d\} = \{s \mid \phi^{-1}(s) \cap Z(\sum_{j=1}^n c_{1,j} y_j - c_{1,n+1}, \dots, \sum_{j=1}^n c_{d,j} y_j - c_{d,n+1}) \text{ is nonempty}\}$$

One should note that Theorem 3.12 applies in this case only because over the canonical base of S, we know that any point on S is of differential transcendence degree 0. So, choosing a generic hyperplane over the base of all of the definable sets at the beginning ensures that the hyperplane remains generic over any given fiber of the definable map ϕ. The set on the right is obviously first order definable (which implies Kolchin constructible by quantifier elimination when K is differentially closed). $\square$

Example 6.2. When S is positive differential transcendence degree, more care is clearly needed, for instance, consider the following example. In the previous theorem, suppose that $S = \mathbb{A}^{n+1}$ and $X \subseteq \mathbb{A}^n$. Let the fiber above a point in S be the hyperplane cut out in $\mathbb{A}^n$ by the coordinates of the point. For instance, fix coordinates

$y_0, \ldots, y_n$ for S, and now fix the system of generic hyperplanes which we propose to use as in the above theorem. Then let $\bar{c}$ be such that $H_{\bar{c}}$ is in the collection. Of course, $\bar{c}$ is a point on S. In the fiber above this point, this hyperplane is not generic, and in fact is precisely the set $\phi^{-1}(\bar{c})$, so intersections with this hyperplane are useless in this fiber. Our solution to this potential problem is completely combinatorial in nature.

Theorem 6.3. *Given a family of differential algebraic quasi-varieties, $\phi : X \to S$, the set $\{s \in S \mid a_m(X_s) = d\}$ is a constructible subset of S.*

Proof. Adopt the notation of Lemma 6.1. Suppose that $a_m(S) = n_1$. Then pick $2n_1 + 1$ systems of $d + 1$ $(n + 1)$-tuples of mutually independent Δ-transcendentals (equivalently, fix an indiscernible set in the generic type, over K with the canonical bases of X, S, and ϕ; then pick any $(2n_1 + 1)(d)(n + 1)$ elements). Denote the chosen elements

$$\{c_{k,i,j} \mid 1 \leq k \leq 2n_1 + 1,\ 1 \leq i \leq d,\ 1 \leq j \leq n + 1\}$$

Of course, over any given fiber of S, some of the $2n_1 + 1$ systems do not determine generic hyperplanes. But, because $a_m(S) = n_1$ and the systems are mutually independent, at least $n + 1$ of the systems are generic over any given fiber $\phi^{-1}(s)$.

Now, the requirement that $a_m(\phi^{-1}(s)) \geq d$ is equivalent to the condition that for at least $n_1 + 1$ values of k,

$$\phi^{-1}(s) \cap Z\Big(\sum_{j=1}^{n} c_{k,1,j} y_j - c_{k,1,n+1}, \ldots, \sum_{j=1}^{n} c_{k,d,j} y_j - c_{k,d,n+1}\Big) \neq \emptyset$$

□

Remark 6.4. The previous theorem tells us some information about the nature of the intersection of our variety V with arbitrary linear subspaces. Namely, that the subset of the Grassmannian which intersects V with fixed dimension n is actually a constructible set in the Kolchin topology.

Note that this would not simply follow from the main generic intersection theorem. After all, there is, a priori, no reason that the exceptional locus could not be an infinite union of differential algebraic subvarieties. In fact, when working with other ranks on differential algebraic varieties, exceptional loci need not have differential algebraic structure. For instance, in [17], a family of differential algebraic varieties in a complex parameter α is shown to be strongly minimal if and only if α is in a certain discrete subset of the real axis (this exceptional locus is far from being a constructible set in the Kolchin topology). The above proposition means that this can not happen when considering the differential dimension.

The authors of [28] prove an interesting geometric result which also generalizes to the partial differential setting. Their proof uses differential specializations. Our approach here is rather different, though a proof by suitably generalized methods of

this kind is possible. Our proof is shorter, but as usual, we are using the machinery of stability theory.

Theorem 6.5. *Let V be a differential algebraic variety of dimension d. If the set of $d+1$ independent generic hyperplanes through $\bar{a}$ intersects V, then $\bar{a} \in V$.*

Proof. We note that the hypotheses imply that $\omega^m \cdot d \leq RU(V/K) < \omega^m \cdot (d+1)$. Let $\bar{a} \notin V$. First, we will argue the result in the case that $\bar{a} = (0, \dots, 0)$. Any hyperplane through the origin is of the the form $\sum c_i y_i = 0$. We assume that the c_i are independent differential transcendentals over K. We denote this hyperplane by $H_{\bar{c}}$. Suppose that $\bar{b}$ is a generic point on one of the irreducible component of $V \cap H_{\bar{c}}$ over $K\langle \bar{c} \rangle$. If $\bar{d}$ is a generic point on V over K, and for $I \subseteq \{1, 2, \dots, n\}$ we have that $D = \{d_i \mid i \in I\}$ is a differential transcendence base for the field extension $K\langle \bar{d} \rangle / K$, then the same property holds for $\bar{b}$, that is $B = \{b_i \mid i \in I\}$ is a differential transcendence base for the field extension $K\langle \bar{b} \rangle / K$. Thus, $RU(\bar{b}/K\langle B \rangle) < \omega^m$.

Now since $\bar{b} \in H_{\bar{c}} \cap V$, we know that $\sum c_i b_i = 0$. We will bound $RU(\bar{b}/K\langle \bar{c} \rangle)$. Since over K, $\bar{c}$ is an independent differential transcendental,

$$RU(\bar{c}/K\langle \bar{b} \rangle) + \omega^m = RU(\bar{c}/K).$$

But, then by Lascar's symmetry lemma [19, chapter 19, for instance]

$$RU(\bar{b}/K\langle \bar{c} \rangle) + \omega^m \leq RU(\bar{b}/K).$$

Thus, the differential transcendence degree of $\bar{b}/K\langle \bar{c} \rangle$ is at least one less than that of $\bar{d}/K$.

In the case that $RU(V/K) < \omega^m$, the above argument using Lascar's symmetry lemma shows that $V \cap H_{\bar{c}} = \emptyset$.

Now, suppose that $\bar{a}$ is some point besides $(0, \dots, 0)$. If so, adjoin $\bar{a}$ to the field K and consider $K\langle \bar{a} \rangle$. A priori, perhaps V is no longer irreducible; if not, arguing about each irreducible component would suffice.

Now, by translating the variety V and the point $\bar{a}$, one can assume $\bar{a} = (0, \dots, 0)$. □

7. Irreducibility in families

In this section, we will produce new results in ordinary differential fields, but we discuss the partial case.

Question 7.1. *Let $\phi : V \to S$ be a morphism of differential algebraic varieties. Is the set $\{s \mid \phi^{-1}(s)$ is irreducible$\}$ a constructible set in the Kolchin topology?*

This question and several other equivalent statements were addressed in [12], but a complete answer to the irreducibility problem was not obtained. We will not address this question of irreducibility in particular, but rather a related one, of which a special case is considered in appendix 3.1 of [6].

Definition 7.2. Let V be a differential algebraic variety. We say that V is *generically irreducible* if V has one component of maximal Kolchin polynomial.

In [6], this notion was considered for finite transcendence degree (in our terminology, dimension zero) ordinary differential algebraic varieties where the following result was proved (in slightly different language):

Proposition 7.3. *Let $\phi : V \to S$ be a morphism of differential algebraic varieties such that the fibers of ϕ are dimension zero. Then the set $\{s \mid \phi^{-1}(s) \text{ is generically irreducible}\}$ is Kolchin-constructible.*

The proof involves two ideas. First, any fiber V_a, being finite rank, is naturally associated with an algebraic D-variety, that is, an algebraic variety W_a along with s_a, a section of the twisted tangent bundle. V_a is *generically* irreducible if and only if W_a is irreducible as an algebraic variety. The construction of W_a from V_a is uniform in families, and irreducibility of W_a is constructible in families [27]. Note that the approach to the problem of irreducibility in families via ultraproducts in [27] (for the algebraic case) was the basic blueprint for the analogous approach in [12] in the differential case where several interesting equivalencies were proved (but the problem was not completely settled). An alternate proof of definability of irreducibility in families of algebraic varieties can be found in [3, 15.5.3].

Example 7.4. Examples of varieties which are generically irreducible, but not irreducible are widely known in the literature. Of course, in the introduction, we gave a difficult example 0.1. Numerous other examples are much easier to develop. For instance, see [11, page 39] or [6, appendix]. For instance, $(\delta^2 x)^2 - 2\delta x$ has components given by $\delta^2 x = 0$ and $\delta^3 x - 1 = 0$.

Proposition 7.5. *Let $\phi : V \to S$ be a morphism of differential algebraic varieties. Then the set $\{s \mid \phi^{-1}(s) \text{ is generically irreducible}\}$ is Kolchin-constructible.*

Proof. The set $\{s \mid dim(\phi^{-1}(s)) = k\}$ is a constructible set by 6.3. So, we need only establish the result for each value of k separately.

For the zero dimensional fibers, we apply 7.3. Suppose that $k > 0$. Recalling the technique and notation in the proof of 6.3, we fix a system of $\{H_{i,j}\}_{i=1,\dots,k,\, j=1,\dots,2dim(S)+1}$ of $2dim(S)+1$ generic independent systems of k hyperplanes. Now, because the intersection of $\phi^{-1}(s)$ with any system of k independent generic hyperplanes over s is dimension zero, if we could fix a single system of k generic independent hyperplanes which were independent from each $s \in S$, we could reduce to the problem to 7.3. Of course, this is impossible when S is not dimension zero. Thus, we must apply the trick from 6.3. A fiber $\phi^{-1}(s)$ of dimension k is generically irreducible if and only if either

(1) $\phi^{-1}(s) \cap H_{1,j} \cap \ldots \cap H_{k,j}$ is zero dimensional and generically irreducible for at least $dim(S)+1$ values of $j \in \{1, \ldots, 2dim(S)+1\}$

(2) For at least $dim(S)+1$ values of $j \in \{1, \dots, 2dim(S)+1\}$, the variety $\phi^{-1}(s) \cap H_{1,j} \cap \dots \cap H_{k,j}$ consists of finitely many points (this implies that the Kolchin polynomial of any component of $\phi^{-1}(s)$ is $d \cdot (t+1)$, which implies that $\phi^{-1}(s)$ is an algebraic variety) and $\phi^{-1}(s)$ is an irreducible algebraic variety

Consider the first case: $\phi^{-1}(s) \cap H_{1,j} \cap \dots \cap H_{k,j}$ being dimension zero is a definable condition; then applying 7.3 for the values of j such that $\phi^{-1}(s) \cap H_{1,j} \cap \dots \cap H_{k,j}$ is zero dimensional gives this is a constructible condition.

Consider the second case: here $\phi^{-1}(s)$ is an algebraic variety, because the differential field generated by a generic point is algebraic over $dim(\phi^{-1}(s)/K\langle s \rangle)$ differential transcendentals. The number of points in the intersection of the variety with $dim(\phi^{-1}(s))$ generic hyperplanes may be bounded in terms of the degree of the equations. In model-theoretic terms, this is known as uniform bounding, and applies in differentially closed fields [11, Corollary 2.15]. So, as s varies, there is a uniform upper bound, n_1. So, checking that there are at least $dim(S)$ such families of hyperplanes which intersect $\phi^{-1}(s)$ in at most n_1 points is a constructible condition. Irreducibility within this constructible subfamily is given by the result of [27] discussed above. $\square$

The problem of irreducibility in families is related to several important problems in differential algebra. In [12], the following theorem is proved:

Theorem 7.6. *The following are equivalent:*

(1) *Irreducibility is definable in families.*
(2) *For every d there exists $r(d, n, m)$ such that for every Δ-field K if P is a prime Δ-ideal of $K\{x\}$ with characteristic set whose elements are of degree and order less that or equal to d, then P is differentially radically generated by Δ-polynomials of order and degree less than or equal to r.*
(3) *For every d, there is $r = r(d, n, m)$ such that for every Δ-field K and all prime Δ-ideals $P, Q \subset K\{x\}$ with characteristic sets whose elements are of degree and order less than or equal to d, if every Δ-polynomial in P of degree and order less than or equal to r is in Q, then $P \subset Q$.*
(4) *For every d, there is $r = r(d, n, m)$ such that for every Δ-field K, every set $S \subset K\{x\}$ of Δ-polynomials of degree and order less than or equal to d, and every pair P and Q of minimal prime Δ-ideals containing S, if every Δ-polynomial in P of degree and order less than or equal to r is in Q, then $P = Q$.*
(5) *For every d, there is $r = r(d, n, m)$ such that for every Δ-field K, every set $S \subset K\{x\}$ of Δ-polynomials of degree and order less than or equal to d, and every $g \in K\{x\}$ of degree and order less than or equal to d, if $gf \notin \{S\}$ for all $f \in \{S\}$ of degree and order less than or equal to r, then g is not a zero divisor modulo $\{S\}$.*

Combining the previous theorem and Proposition 7.5, we obtain:

Proposition 7.7. *(In the ordinary case) Each of the conditions of the previous theorem are equivalent to the following:*

- *Let $\phi : V \to S$ be a morphism of differential algebraic varieties. The set*
$$\{s \mid \phi^{-1}(s) \text{ is generically irreducible, but not irreducible}\}$$
a constructible set in the Kolchin topology.

Remark 7.8. As we noted at the beginning of this section, we are working only in ordinary differential fields for this application; the results and approach discussed here do not seem to readily apply to the partial differential case. One might still apply the intersection theory developed above to reduce the question of generic irreducibility to the dimension zero case. However, difficulties still abound, because the Δ-*type* of the variety is likely to be greater than zero. In this case, one can not reduce to the algebraic category via the functor to algebraic D-varieties, but must use the prolongation sequences of [14].

References

[1] James Freitag. Completeness in partial differential algebraic geometry. *Under Review, Journal of Algebra, http://arxiv.org/abs/1106.0703*, 2011.

[2] James Freitag. Generics in differential fields. *In Preparation, http://www.math.uic.edu/ freitag/GenericPoints.pdf*, 2011.

[3] A. Grothendieck. *Eléments de géométrie algérique (rédigér avec la collaboration de Jean Dieudonné): IV. Étude locale des schémas et des morphismes de schémas*, volume 32. Publications Mathématiques de l'IHÉS, 1967.

[4] Robin Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathemematics 52. Springer-Verlag, 1977.

[5] W.V.D. Hodge and D. Pedoe. *Methods of Algebraic Geometry*. Number v. 2 in Cambridge Mathematical Library. Cambridge University Press, 1994.

[6] Ehud Hrushovski and Masanori Itai. On model complete differential fields. *Transactions of the American Mathematical Society*, Volume 355(11):4267–4296, 2003.

[7] Ellis Kolchin. Differential equations in a projective space and linear dependence over a projective variety. *Contributions to analysis (A collection of papers dedicated to Lipman Bers*, pages 195–214, 1974.

[8] Ellis R. Kolchin. *Differential Algebra and Algebraic Groups*. Academic Press, New York, 1976.

[9] Ellis R. Kolchin. *Differential Algebraic Groups*. Academic Press, New York, 1984.

[10] David Marker. *Model Theory: an Introduction*. Springer, Graduate Texts in Mathematics, 217, Second Edition, 2002.

[11] David Marker, Margit Messmer, and Anand Pillay. *Model theory of fields*. A. K. Peters/CRC Press, 2005.

[12] Rahim Moosa Matthew Harrison-Trainor, Jack Klys. Nonstandard methods for bounds in differential polynomial rings. *Preprint*, page http://www.math.uwaterloo.ca/ rmoosa/nsda7feb2012.pdf, 2011.

[13] Tracy McGrail. The model theory of differential fields with finitely many commuting derivations. *Journal of Symbolic Logic*, 65, No. 2:885–913, 2000.

[14] Rahim Moosa, Anand Pillay, and Thomas Scanlon. Differential arcs and regular types in differential fields. *Journal fur die reine and angewandte Mathematik*, 620:35–54, 2008.

[15] Rahim Moosa and Thomas Scanlon. Jet and prolongation spaces. *Journal de l'Institut de Mathematiques de Jussieu*, 9(2):391–430, 2010.
[16] Anand Pillay. *Geometric Stability Theory.* Oxford University Press, 1996.
[17] Anand Pillay and Ronnie Nagloo. On algebraic relations between solutions of a generic Painleve equation. *Preprint, http://arxiv.org/abs/1112.2916v1*, 2011.
[18] Anand Pillay and Martin Ziegler. Jet spaces of varieties over differential and difference fields. *Selecta Math. (N. S.)*, 9 (4):579–599, 2003.
[19] Bruno Poizat. *A course in model theory.* Springer, New York, 2000.
[20] Wai Yan Pong. Complete sets in differentially closed fields. *Journal of Algebra*, 224:454–466, 2000.
[21] Wai Yan Pong. Some applications of ordinal dimensions to the theory of differentially closed fields. *Journal of Symbolic Logic*, 65, No. 1:347–356, 2000.
[22] Joseph Fels Ritt. *Differential Algebra.* Dover Publications, New York, 1950.
[23] William Sit. Anomaly of intersections in differential algebra. *Online notes, http://www.sci.ccny.cuny.edu/~ksda/PostedPapers/Sit021012.pdf.*
[24] William Sit. The Ritt-Kolchin theory for dierential polynomials. *Dierential Algebra and Related Topics, Proceedings of the International Workshop, NJSU*, 23 November 2000.
[25] Sonat Suer. *Model theory of differentially closed fields with several commuting derivations.* PhD thesis, University of Illinois at Urbana-Champaign, 2007.
[26] Sonat Suer. On subgroups of the additive group in differentially closed fields. *Journal of Symbolic Logic*, 77:369–391, 2012.
[27] Lou van den Dries. Bounds in the theory of polynomial rings over fields. *Inventiones Mathematicae*, 76:77–91, 1984.
[28] C. M. Yuan X. S. Gao, W. Li. Intersection theory in dierential algebraic geometry: Generic intersections and the dierential chow form. *Accepted by Trans. of of Amer. Math. Soc., 58 pages. Also in arXiv:1009.0148v1*, 2011.

freitag@math.berkeley.edu, Department of Mathematics, University of California, Berkeley, 970 Evans Hall, Berkeley, CA 94720-3840